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Quantum Logic Axioms and the Proposition-State Structure†

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Abstract

A possible picture of the axiomatic basis of quantum mechanics is drawn and the set of propositions of the quantum logic approach to quantum mechanics is shown to be a complete, orthocomplemented and weakly modular lattice. A condition that the set of propositions be atomic is found, in which the notion of 'characteristic state' is involved. This scheme is compared with the usual Hilbert one, and in a Hilbert picture in which discrete superselection rules can be present also, the characteristic states are shown to be the pure states.

1. *Introduction*

In the quantum logic approach to the foundations of quantum mechanics, the set of propositions—or, as they are also called, experimental questions or events—is assumed to have a definite mathematical structure (Jauch, 1968; Varadarajan, 1968). For instance, in Piron's formulation the set of propositions is assumed to be a 'generalised proposition system' (Piron, 1964). A motivation for such an axiomatic structure is that the usual Hilbert model of quantum mechanics can be in one sense obtained from it (Piron, 1964; Amemiya & Araki, 1967; Gudder & Piron, 1971). Moreover, the quantum logic axioms have been related by Plymen (1968a, b) with the C^*-algebra approach to quantum mechanics (Segal, 1947; Haag & Kastler, 1964; Kadison, 1965). Finally, we remark that the atomicity is a request for the set of propositions which is not embodied in the axioms that define a generalised proposition system. The atomicity is anyway usually assumed in order to obtain quantum mechanics in Hilbert space.

The aforementioned quantum logic axiomatic structure has been supported by Jauch and Piron in a picture in which only the propositions are

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considered to be the basic constituents, while the states are introduced just as particular sets of propositions. This notion of state applies to individual systems, while the usual quantum mechanical notion of state is a statistical concept. The price to be paid for such a picture is the introduction of a notion of 'true', which is rather an intuitive than a definite concept, since in this scheme no solution is given to the question 'how to produce systems for which a given yes–no experiment is known to be 'true' or how to obtain this knowledge' (Jauch & Piron, 1969).

In this paper we present a reformulation of these ideas according to a scheme in which propositions and states are equally primitive entities. The dual natures of the concepts of state and proposition are in fact apparent from an operational viewpoint, since a state can be identified in one sense by the set of values taken in it by the set of all the propositions and a proposition can be defined in terms of the set of all the states (Gunson, 1967). In our framework the notion of state is, anyway, more general than the usual one, since σ-additivity is not assumed. This makes sense, as we are not concerned with observables, and hence with the problem of obtaining probability measures on the real line from states and observables.

According to this picture we shall introduce a triple which will be called proposition-state structure and which can be interpreted in a plausible physical way. Assuming therein three physically sensible axioms, we can show in Section 2 that the set of propositions is a generalised proposition system. After introducing the notion of 'characteristic state', in Section 3 we find a condition for the proposition-state structure from which the atomicity of the set of propositions derives. The condition for the atomicity can be interpreted as the existence of 'sufficiently many' characteristic states. Finally, in Section 4 we show that in the usual Hilbert model the characteristic states are right the pure states; this is true also when discrete superselection rules are present. Hence in this model we get a one-to-one correspondence from the set of states onto the set of atoms, while this correspondence has not to be a bijection as a general rule, even if in the set of propositions the atomicity holds.

2. *The Proposition-State Structure*

We assume that, if we are given a physical system, then we have a triple (L, S, P) which admits the following phenomenological interpretation. The term L denotes a set, the elements of which are the observation procedures which admit only two possible results, say 'yes' or 'no', namely instructions for an apparatus which interacts with a sample of the physical system and indicates either 'yes' or 'no' in correspondence with the 'occurrence' or 'non-occurrence' of a particular phenomenon pertaining to the physical system. The term S also denotes a set, the elements of which are called states. A state may be identified with a preparation procedure, namely instructions for an apparatus which produces samples of the physical system. The term P stands for a function $P: L \times S \to [0,1]$. The interpretation of $P(\alpha, s)$ for $\alpha \in L$ and $s \in S$ can be stated in the following way. Prepare an ensemble of samples of the physical system according to the procedure s. Determine whether 'yes' or 'no' occurs, observing by α each sample of the ensemble. Taking the number of the samples of the ensemble sufficiently large, the frequency of 'yes' can be made as close to $P(\alpha, s)$ as one wants.

If $\alpha \in L$, then the subsets $S_1(\alpha)$ and $S_0(\alpha)$ are defined to be $S_1(\alpha) \equiv \{s \in S; P(\alpha, s) = 1\}$ and $S_0(\alpha) \equiv \{s \in S; P(\alpha, s) = 0\}$. Let us now define on L the relation R as follows: if α, $\beta \in L$, $\alpha R \beta$ iff $S_1(\alpha) = S_1(\beta)$. It is then easy to show that R is an equivalence relation and we can define the set of equivalence classes $\mathscr{L} \equiv L/R$; $[\alpha]$ is the equivalence class containing the element $\alpha \in L$. We can define on $\mathscr{L}$ the relation $\leqslant$ as follows: if $[\alpha]$, $[\beta] \in \mathscr{L}$, $[\alpha] \leqslant [\beta]$ iff $S_1(\alpha) \subseteq S_1(\beta)$.

Proposition 2.1. The set $\mathscr{L}$ with the relation $\leqslant$ is a complete lattice, that is $\leqslant$ is a partial ordering on $\mathscr{L}$ and for any family $\{[\alpha_k]\}(k \in K)$ of elements of $\mathscr{L}$ with index set K, both the least upper bound and the greatest lower bound exist.

Proof. The relation $\leqslant$ is evidently reflexive and transitive, since the set-theoretical inclusion $\subseteq$ is a reflexive and transitive relation. The definition of R and the antisymmetry of $\subseteq$ imply that $\leqslant$ is antisymmetric. Hence the relation $\leqslant$ is a partial ordering on $\mathscr{L}$.

If $\{[\alpha_k]\}(k \in K)$ is any family of elements of $\mathscr{L}$ with index set K, we can construct an element of L defining an observation procedure in this way: choose at random one of the observation procedures of the family $\{\alpha_k\}(k \in K)$, and then use its apparatus to get the outcome 'yes' or 'no'. The element of L which has now been defined will be denoted by $\prod_k \alpha_k$. It is now easy to show that the element $[\prod_k \alpha_k]$ of $\mathscr{L}$ depends only on the family $\{[\alpha_k]\}(k \in K)$ of elements of $\mathscr{L}$ and not on the family $\{\alpha_k\}(k \in K)$ of elements of L which has been used to define $\prod_k \alpha_k$. From the definition of $\prod_k \alpha_k$ we have in fact

$$S_1(\prod_k \alpha_k) = \bigcap_k S_1(\alpha_k).$$

Hence if $\{\alpha_k'\}(k \in K)$ is a family of elements of $\mathscr{L}$ such that $\alpha_k R \alpha_k'$, $\forall k \in K$, it follows that

$$S_1(\prod_k \alpha_k) = \bigcap_k S_1(\alpha_k) = \bigcap_k S_1(\alpha_k') = S_1(\prod_k \alpha_k'),$$

which is equivalent to $\prod_k \alpha_k R \prod_k \alpha_k'$. If $[\beta]$ is an element of $\mathscr{L}$, then $[\beta] \leqslant [\alpha_k]$, $\forall k \in K$ iff $S_1(\beta) \subseteq S_1(\alpha_k)$, $\forall k \in K$ iff

$$S_1(\beta) \subseteq \bigcap_k S_1(\alpha_k) = S_1(\prod_k \alpha_k)$$

iff $[\beta] \leqslant [\prod_k \alpha_k]$. Hence the greatest lower bound exists for the family

$\{[\alpha_k]\}(k \in K)$ and it is $[\prod_k \alpha_k]$. We shall denote it by $\bigwedge_k [\alpha_k]$, since we reserve the symbol $\cap$ for the set-theoretical meet.

It is now easy to show that for the family $\{[\alpha_k]\}(k \in K)$ also the least upper bound exists and that it is the greatest lower bound of the family $\{[\beta] \in \mathscr{L};\ [\alpha_k] \leqslant [\beta], \forall k \in K\}$. It will be denoted by $\bigvee_k [\alpha_k]$, since we reserve the symbol $\cup$ for the set-theoretical join. The least and the greatest elements of $\mathscr{L}$ will be denoted by ϕ and I respectively. They contain the trivial observation procedures $\hat{\phi}$ and $\hat{I}$ defined by saying 'no' or 'yes' respectively, simply because the physical system exists. The set $S_1(\hat{I})$ is in fact S and the set $S_1(\hat{\phi})$ is the empty set of states.

The theorem which we have now proved is the counterpart in (L,S,P) of a theorem which can be found in the paper of Jauch & Piron (1969). $\mathscr{L}$ will be called the logic of the physical system and the elements of $\mathscr{L}$ propositions, following a terminology introduced by Birkhoff & von Neumann (1936). According to the phenomenological interpretation of the function P, two observation procedures α, $\beta \in L$ are distinguishable in our framework only if a preparation procedure $s \in S$ exists such that $P(\alpha,s) \neq P(\beta,s)$. We want, however, two equivalent elements of L to be undistinguishable, because we want to consider as basic elements of our picture the propositions, namely the elements of $\mathscr{L}$. This is the motivation to state our first axiom.

Axiom 1. If α and β are elements of L, then from $\alpha R \beta$ it follows that $P(\alpha,s) = P(\beta,s)$, $\forall s \in S$.

From Axiom 1 it follows that P can be considered to be a function on $\mathscr{L} \times S$ rather than on $L \times S$, assuming $P([\alpha],s) = P(\alpha,s)$ if $\alpha \in L$ and $s \in S$. Accordingly, $S_1(a)$ and $S_0(a)$ can be re-defined in an obvious way if $a \in \mathscr{L}$. Hence trivially we get

$$S_1(\bigwedge_k a_k) = \bigcap_k S_1(a_k)$$

for any family $\{a_k\}(k \in K)$ of elements of $\mathscr{L}$ with index set K, $S_1(a) = S$ iff $a = I$, $S_1(a)$ is the empty set of states iff $a = \phi$. The triple $(\mathscr{L},S,P)$ will be called proposition-state structure, following a terminology introduced by Pool (1968).

If α is an observation procedure, then we can define another element $\tilde{\alpha}$ of L, simply interchanging the labels of the outcomes 'yes' and 'no' in the apparatus which defines α. From the very definitions and from the phenomenological interpretation of P it follows that, if $\alpha \in L$, then $P(\tilde{\alpha},s) = 1 - P(\alpha,s)$, $\forall s \in S$. Hence we get $S_1(\alpha) = S_0(\tilde{\alpha})$ and $S_1(\tilde{\alpha}) = S_0(\alpha)$, $\forall \alpha \in L$, $\forall s \in S$. As a consequence of Axiom 1, we can now define the mapping

$$c\colon \mathscr{L} \to \mathscr{L},\ c[\alpha] = [\tilde{\alpha}].$$

Indeed, if two elements α and β of L are such that $\alpha R \beta$, then $S_0(\alpha) = S_0(\beta)$ holds, from which it follows that $S_1(\tilde{\alpha}) = S_1(\tilde{\beta})$. Hence we get $\tilde{\alpha} R \tilde{\beta}$.

If $a \in \mathscr{L}$, from $S_1(a \wedge ca) = S_1(a) \cap S_0(a)$ it follows that $S_1(a \wedge ca)$ is the empty set of states. Hence we get $a \wedge ca = \phi$ and ca could be a complement of a. The proposition ca would in fact result in a complement of a if $S_1(a \vee ca)$ could be shown to be the full set of states in a way similar to that followed previously for $S_1(a \wedge ca)$. But this is impossible, because the relations $S_1(a \vee b) = S_1(a) \cup S_1(b)$ for any a and b in $\mathscr{L}$ and $S_1(ca) = S - S_1(a)$ for any $a \in \mathscr{L}$ hold only if we consider for the physical system a classical system and for S the set of pure states, as will be pointed out later. Otherwise, both equalities cannot be asserted to be true. Hence we have to postulate that the pair of propositions a and ca behave in a 'classical' way for any $a \in \mathscr{L}$; namely, we have to state the following axiom.

Axiom 2. If a is any proposition of $\mathscr{L}$, then $S_1(a \vee ca) = S$.

From Axiom 2 it follows that $a \vee ca = I$ for any $a \in \mathscr{L}$. Hence ca is a complement of a.

Our picture will be complete if we assume our last axiom.

Axiom 3. If a and b are two propositions of $\mathscr{L}$ such that $S_1(a) \subseteq S_1(b)$ then the sublattice of $\mathscr{L}$ generated by the family $\{a, ca, b, cb\}$ is distributive.

We can now in fact show the main result of this section.

Proposition 2.2. The logic $\mathscr{L}$ is a generalised proposition system in the sense of Piron (1964).

Proof. In Proposition 2.1 we have shown that $\mathscr{L}$ with the relation $\leqslant$ is a complete lattice, hence the Axioms O and T of Piron are satisfied by $\mathscr{L}$. If a and b are two elements of $\mathscr{L}$ such that $a \leqslant b$, then—taking into account Axioms 2 and 3 and that from $a \leqslant b$ it follows that $cb \wedge a \leqslant cb \wedge b = \phi$—we get $cb = cb \wedge (a \vee ca) = (cb \wedge a) \vee (cb \wedge ca) = cb \wedge ca$, whence $cb \leqslant ca$ follows. Moreover, for any $a \in \mathscr{L}$, $c(ca) = a$ holds because of the very definition of c, along with $a \wedge ca = \phi$, which has already been proved. Hence c is an orthocomplementation and Piron's Axiom C holds in L. Then Axiom 3 results in Piron's Axiom P exactly, and this completes the proof of the theorem.

The request that Axioms 1, 2 and 3 hold in the proposition-state structure $(\mathscr{L}, S, P)$ amounts therefore to require that the Axioms O, T, C, P hold for the set $\mathscr{L}$ of all the propositions of the physical system. We notice that, after Proposition 2.2, Axiom 3 results in nothing else than the axiom of weak-modularity, which has been widely motivated, e.g. by Piron (1964) and by Jauch (1968).

If $s \in S$ then, as a consequence of Axiom 1, we can define the function $\hat{s}: \mathscr{L} \to [0,1]$, $\hat{s}(a) = P(a,s)$. The set $\hat{S}$ of all the functions $\hat{s}$ has the properties:

(a) $\hat{s}(\hat{I}) = 1$ and $\hat{s}(\phi) = 0$, $\forall \hat{s} \in \hat{S}$;
(b) $\hat{s}(a) = 1$ iff $\hat{s}(ca) = 0$, $\forall \hat{s} \in \hat{S}$, $\forall a \in \mathscr{L}$;

(c) if $a,b \in \mathscr{L}$ and $\hat{s} \in \hat{S}$ then $\hat{s}(a) = 1 \Rightarrow \hat{s}(b) = 1$ iff $a \leqslant b$;
(d) if $a \in \mathscr{L}$ then $a \neq \phi$ iff $\exists \hat{s} \in \hat{S}$, $\hat{s}(a) = 1$;
(e) if $a,b \in \mathscr{L}$ then $a \neq b$ iff $\exists \hat{s} \in \hat{S}$, $\hat{s}(a) \neq \hat{s}(b)$.

All these assertions follow directly from the definition of $\hat{S}$. The set S of states is called convex if the following property holds. Let s_1 and s_2 be two elements of S and k a rational number such that $0 < k < 1$. Then there is an element s of S such that $\hat{s} = k\hat{s}_1 + (1-k)\hat{s}_2$. The element s of S which defines $\hat{s}$ will be denoted by the symbol $ks_1 + (1-k)s_2$, and it can be easily interpreted as follows. Let N be any integer such that kN is an integer. If you prepare N samples of the physical system by the procedure $ks_1 + (1-k)s_2$, then you have to prepare kN of them by the procedure s_1 and the others by the procedure s_2. For convenience, in the sequel S will be assumed to be convex. This property will be used only to show Proposition 3.3 in the next section.

Finally, it should be stressed that we have had to introduce Axiom 2 because in a general proposition-state structure the usual logical significance of the lattice operations does not hold. In fact, while the proposition $a \wedge b$ is true iff both a and b are true since $S_1(a \wedge b) = S_1(a) \cap S_1(b)$ holds by definition, we cannot assert that the proposition $a \vee b$ is true iff a or b, or both, are true. Indeed, $S_1(a) \cup S_1(b) \subseteq S_1(a \vee b)$ holds because $a \leqslant a \vee b$ and $b \leqslant a \vee b$, but $S_1(a) \cup S_1(b) = S_1(a \vee b)$ does not hold as a rule, as it is shown by examples which can be easily constructed in quantum mechanics as well as in classical mechanics, if for the classical case we also consider in S states which are not pure. The equality holds also for the join, and the usual logical interpretation of both the meet and the join is then possible, if the physical system is a classical one and if in S we consider pure states only. In this case the lattice is in fact the power set of the phase space Ω and a state is defined fixing a point of Ω. If $a \in \mathscr{L}$, the state s is then in $S_1(a)$ iff $s \in a$. Hence $S_1(a \vee b) = S_1(a) \cup S_1(b)$ trivially holds, since both sets are in fact the subset $a \cup b$ of Ω.

An analogous discussion could be performed about the relation $S_1(ca) = S - S_1(a)$, which does not hold but for the case of a classical system with only pure states.

3. *The Atomicity Condition*

The atomicity condition is generally postulated for the logic $\mathscr{L}$ in order to get the usual Hilbert space model for quantum mechanics. Then we want to find a possible relationship between the atomicity of $\mathscr{L}$ and some features of the proposition-state structure $(\mathscr{L}, S, P)$. First of all, we shall find a strict correlation between atoms and a class of subsets of $\mathscr{L}$.

Definition 3.1. A subset $\mathscr{I}$ of $\mathscr{L}$ is called an ideal if the following relations hold:

(a) $\phi \notin \mathscr{I}$;
(b) $a \vee b \in \mathscr{I}, \forall a \in \mathscr{I}, \forall b \in \mathscr{L}$;
(c) if $\{a_k\}(k \in K)$ is a family of elements of $\mathscr{L}$ with index set K, then $a_k \in \mathscr{I}, \forall k \in K \Rightarrow \bigwedge_k a_k \in \mathscr{I}$.

If in (c) we assume the index set to be finite, then we get what is called by Birkhoff (1967) a dual ideal. Notice that condition (b) in Definition 3.1 is equivalent to $a \in \mathscr{I}, b \in \mathscr{L}, a \leqslant b \Rightarrow b \in \mathscr{I}$. The notion of ideal being very important in what follows, we characterise it by means of the following theorem.

Proposition 3.1. If $\mathscr{I} \subseteq \mathscr{L}$ then $\mathscr{I}$ is an ideal in $\mathscr{L}$ iff $\exists a \in \mathscr{L}, a \neq \phi$ such that $\mathscr{I} = \{x \in \mathscr{L}; a \leqslant x\}$. Then $a = \bigwedge_{x \in \mathscr{I}} x \equiv \bigwedge \mathscr{I}$.

Proof. It is trivial.

An ideal is called maximal if it is a maximal element in the set of all the ideals of $\mathscr{L}$ ordered with respect to set inclusion. We notice that the existence of maximal ideals cannot be proved by a Zorn-like argument because of condition (c) in Definition 3.1. We shall now state the correlation between atoms and maximal ideals.

Propositon 3.2. If $\mathscr{I}$ is an ideal in $\mathscr{L}$ and $a = \bigwedge \mathscr{I}$, then the following conditions are equivalent:

(a) $\mathscr{I}$ is maximal, namely for an ideal $\mathscr{I}'$ in $\mathscr{L}$, $\mathscr{I} \subseteq \mathscr{I}' \Rightarrow \mathscr{I} = \mathscr{I}'$;
(b) a is an atom, namely $a \neq \phi$ and for an element $b \in \mathscr{L}$ different from a, $b \leqslant a \Rightarrow b = \phi$.

Proof. (a) $\Rightarrow$ (b): $a \neq \phi$ follows from (a) and (c) of Definition 3.1. If $b \neq a$ could exist in $\mathscr{L}$ such that $b \leqslant a$ and $b \neq \phi$, then for $\mathscr{I}' \equiv \{x \in \mathscr{L}; b \leqslant x\}$ it would be seen to hold $\mathscr{I} \subseteq \mathscr{I}'$ along with $\mathscr{I} \neq \mathscr{I}'$: apply Proposition 3.1 and notice that $b \in \mathscr{I}'$ and $b \notin \mathscr{I}$. Hence $\mathscr{I}$ fails to be maximal.

(b) $\Rightarrow$ (a): if $\mathscr{I}'$ is an ideal such that $\mathscr{I} \subseteq \mathscr{I}'$, then we get $\phi \neq \bigwedge \mathscr{I}' \leqslant \bigwedge \mathscr{I}$, from which $\bigwedge \mathscr{I}' = \bigwedge \mathscr{I}$ follows because $\bigwedge \mathscr{I}$ is an atom. From Proposition 3.1 we get $\mathscr{I} = \mathscr{I}'$.

Let us now introduce ideals in $\mathscr{L}$ using the proposition-state structure. If $s \in S$, define $\mathscr{I}_s \equiv \{x \in \mathscr{L}; s \in S_1(x)\}$. Whatever the state s is, $\mathscr{I}_s$ cannot be the empty set of the set $\mathscr{L}$, since $I \in \mathscr{I}_s, \forall s \in S$. Taking into account that $S_1(\phi)$ is the empty set of states, that $a \leqslant b$ is equivalent to $S_1(a) \subseteq S_1(b)$, and that

$$S_1(\bigwedge_k a_k) = \bigcap_k S_1(a_k),$$

we can show that $\mathscr{I}_s$ is an ideal.

If a state is thoroughly characterised by the propositions which are true in it then it singles out an atom, as we shall presently see. First we need a definition.

Definition 3.2. We say that a state s is characteristic if $s' \in S$, $\mathscr{I}_{s'} = \mathscr{I}_s \Rightarrow s' = s$.

We denote the set of characteristic states by S_c. The next theorem shows how maximal ideals are related to characteristic states.

Proposition 3.3. If $s \in S_c$, then $\mathscr{I}_s$ is a maximal ideal.

Proof. Let us suppose that $\mathscr{I}_s$ is not a maximal ideal. Then s will be proved not to be characteristic. Indeed if $\mathscr{I}_s$ is not maximal then, because of Proposition 3.2, $\wedge \mathscr{I}_s$ is not an atom and this in turn implies that $b \in \mathscr{L}$ exists such that $\phi \leqslant b \leqslant \wedge \mathscr{I}_s$ along with $b \neq \phi$ and $b \neq \wedge \mathscr{I}_s$. Hence a state s' exists such that $s' \in S_1(b)$, since $b \neq \phi$, and $s' \neq s$, since from $b \neq \wedge \mathscr{I}_s$ it follows that $b \notin \mathscr{I}_s$, and this in turn implies that $s \notin S_1(b)$. Because of convexity of S, a state $\bar{s}$ exists such that $\bar{s} = ks' + (1-k)s$, where k is a rational such that $0 < k < 1$. We can show that $\bar{s} \neq s$ observing that $P(b,\bar{s}) = P(b,s)$ holds iff $P(b,s') = P(b,s)$ holds, but $P(b,s') = 1$ while $P(b,s) \neq 1$. It is now easy to prove that, for any proposition $x \in \mathscr{L}$, $P(x,\bar{s}) = 1$ iff $P(x,s') = P(x,s) = 1$. Hence we get $\mathscr{I}_{\bar{s}} = \mathscr{I}_{s'} \cap \mathscr{I}_s$. From $b \leqslant \wedge \mathscr{I}_s$, $S_1(b) \subseteq S_1(\wedge \mathscr{I}_s)$ follows. Thus s' belongs to $S_1(\wedge \mathscr{I}_s)$ and consequently $\mathscr{I}_s$ is a subset of $\mathscr{I}_{s'}$. In this way we have found that for $\bar{s}$ the two relations $\bar{s} \neq s$ and $\mathscr{I}_{\bar{s}} = \mathscr{I}_s$ are fulfilled, namely that s is not characteristic. This proves the theorem.

We can now easily show the main theorem of this section.

Proposition 3.4. Let the logic $\mathscr{L}$ be such that for any proposition $a \in \mathscr{L}$, $a \neq \phi$, at least one of the following conditions holds:

(a) a is an atom;
(b) $S_1(a) \cap S_c$ is not the empty set of states.

Then $\mathscr{L}$ is an atomic lattice, namely for any $a \in \mathscr{L}$, $a \neq \phi$, this condition holds:

(A) an atom p exists such that $p \leqslant a$.

Proof. Let $a \in \mathscr{L}$, $a \neq \phi$, be such that condition (a) holds. Then condition (A) is true with $p = a$. Take now $a \in \mathscr{L}$, $a \neq \phi$, such that condition (b) holds. If $s \in S_1(a) \cap S_c$ then $a \in \mathscr{I}_s$ and $\mathscr{I}_s$ is a maximal ideal, as a consequence of Proposition 3.3. Hence, taking $p = \wedge \mathscr{I}_s$, we get condition (A), because $p \leqslant a$ and p is an atom, as a consequence of Proposition 3.2.

This theorem provides a condition for the proposition-state structure which assures the atomicity of $\mathscr{L}$. The meaning of this condition is to require the existence of 'sufficiently many' characteristic states.

Proposition 3.3 sets up a correspondence from the set of characteristic states into the set of maximal ideals: if s is a characteristic state, a maximal ideal $\mathscr{I}$ exists such that $s \in S_1(\wedge \mathscr{I})$. It is in fact $\mathscr{I}_s$. It should be noted that, in the general picture which we have drawn, this correspondence need not be a bijection even if the lattice is atomic. Let $\mathscr{I}$ be in fact a maximal ideal and $p = \wedge \mathscr{I}$. From Proposition 3.2 it follows that p is an atom. Then, without any contradiction with the condition of Proposition 3.4, for the atom p the set of states $S_1(p) \cap S_c$ could happen to be the empty set. Hence no characteristic state s can exist such that $s \in S_1(\wedge \mathscr{I})$ and the correspondence from the set of characteristic states into the set of maximal ideals is not onto. We notice that, if $S_1(p) \cap S_c$ is empty, there are infinitely many states s (none of them can be characteristic!) for which $s \in S_1(p)$ (whence, by maximality of $\mathscr{I}$, we get also $\mathscr{I}_s = \mathscr{I}$). Indeed, since p is different from ϕ, a state $\bar{s}$ must exist in $S_1(p)$. At least another state s' must then exist in $S_1(p)$ [if $S_1(p)$ would in fact contain only the state $\bar{s}$, then $\bar{s}$ could be easily shown to be characteristic]. Hence for any state $s = k\bar{s} + (1-k)s'$, with k rational and $0 \leqslant k \leqslant 1$, we get $s \in S_1(p)$.

4. *The Hilbert Model*

We can now ask what happens to the picture discussed in Section 3 when the usual Hilbert model is assumed. First we give here a brief account of this model, while a detailed description can be found elsewhere (Cirelli & Gallone, 1972).

When in the Hilbert model the superselection rules are assumed to be discrete, then the logic $\mathscr{L}$ is taken to be a direct union of standard logics, namely

$$\mathscr{L} = \bigvee_{k \in K}^{\oplus} \mathscr{L}_k$$

where K is a finite or countable index set and the standard logic $\mathscr{L}_k$ is the orthocomplemented lattice of all the projections of a complex separable Hilbert space $\mathscr{H}_k$ (Varadarajan, 1968). For each $k \in K$, a 'projection' π_k is defined from $\mathscr{L}$ onto $\mathscr{L}_k$ as the function which maps any element of $\mathscr{L}$ into its component in $\mathscr{L}_k$. The projection π_k is a homomorphism, namely it preserves the meet, the join and the orthocomplementation (Gallone & Manià, 1971). The set of states (we call here state and denote by s the function $\hat{s}$ which is defined by a state s as in Section 2) is taken in this scheme to be the set of all the positive, σ-additive functions s on $\mathscr{L}$ such that $S(I) = 1$, where I is the unit element of $\mathscr{L}$. We recall that σ-additivity means

$$s(\bigvee_n a_n) = \sum_n s(a_n)$$

if $a_n \leqslant ca_m$ for $n \neq m$, where $\{a_n\}$ is a countable family of elements of $\mathscr{L}$. We can now easily prove that a function s on $\mathscr{L}$ is a state if and only if there is a sequence $\{x_n\}$ of vector fields $x_n : K \ni k \to x_n(k) \in K$ such that the vectors of the family $\{x_n(k)\}$ are orthogonal for each $k \in K$, the condition

$$\sum_k \sum_n \|x_n(k)\|^2 = 1 \tag{4.1}$$

holds and the equality

$$s(a) = \sum_k \sum_n (x_n(k), \pi_k(a) x_n(k)), \qquad \forall a \in \mathscr{L} \tag{4.2}$$

is satisfied.

In fact, if for a function s the relation (4.2) holds, then it is trivially positive and σ-additive, and from the condition (4.1) we get $s(I) = 1$. Hence it is a state. Conversely, if s is a state, then for each $k \in K$ we can define on $\mathscr{L}_k$ the function

$$s_k : \mathscr{L}_k \to [0, 1], s_k(\alpha) = \begin{cases} 0 & \text{if } s(\pi_k(I)) = 0, \\ s(\alpha).(s(\pi_k(I)))^{-1} & \text{if } s(\pi_k(I)) \neq 0. \end{cases}$$

When s_k is not the null function, then it is easily shown to be a positive σ-additive function on the standard logic $\mathscr{L}_k$ such that $s_k(\pi_k(I)) = 1$, where $\pi_k(I)$ is the unit element of $\mathscr{L}_k$ (namely it is the unit operator on $\mathscr{H}_k$). Hence, by a theorem of Gleason (1957) for each $k \in K$ a sequence of orthogonal vectors $\{y_n(k)\}$ in $\mathscr{H}_k$ exists such that

$$s_k(\alpha) = \sum_n (y_n(k), \alpha y_n(k)), \qquad \forall \alpha \in \mathscr{L}_k$$

[if s_k is the null function, the sequence is simply constructed taking $y_n(k)$ to be the zero vector for any index n]. Define then the sequence of vector fields $\{x_n\}$ such that $x_n(k) = y_n(k).(s(\pi_k(I)))^{1/2}$. The condition (4.1) holds for $\{x_n\}$, since

$$\sum_k \sum_n \|x_n(k)\|^2 = \sum_k \sum_n (y_n(k), \pi_k(I)\, y_n(k)).s(\pi_k(I))$$
$$= \sum_k s(\pi_k(I)) = s(\bigvee_k \pi_k(I)) = 1$$

Moreover the state s is connected with the sequence $\{x_n\}$ through the relation (4.2). Indeed for any element $a \in \mathscr{L}$ we get

$$s(a) = s(\bigvee_k \pi_k(a))$$
$$= \sum_k s(\pi_k(a)) = \sum_k s_k(\pi_k(a)).s(\pi_k(I))$$
$$= \sum_k \sum_n (x_n(k), \pi_k(a)\, x_n(k))$$

We point out that the scheme of the Hilbert model is less general than the framework of the previous sections, not only because $\mathscr{L}$ is now made up by sequences of projections but also because the states are now taken to be σ-additive. This is in fact a very strong property which has not been assumed for the states in Sections 2 and 3.

We shall now show that in the Hilbert model a state s is characteristic iff it is represented by a ray. Hence in this model the characteristic states are in fact the pure states because of Theorem 7.23 of Varadarajan's book (1968). Before turning to the proof of this property of characteristic states in the Hilbert model, we need to find in it a condition under which two states s and s' define the same ideal. Let $\{x_n\}$ and $\{y_m\}$ be two sequences of vector fields related to the states s and s' respectively, as in (4.2). Then we have

$$\mathscr{I}_s = \mathscr{I}_{s'} \text{ iff } \underset{n}{\sigma}\{x_n(k)\} = \underset{m}{\sigma}\{y_m(k)\}, \qquad \forall k \in K \tag{4.3}$$

where $\underset{n}{\sigma}\{x_n(k)\}$ and $\underset{m}{\sigma}\{y_m(k)\}$ are the closed linear span in $\mathscr{H}_k$ of the families of vectors $\{x_n(k)\}$ and $\{y_m(k)\}$ respectively. Being a an element of $\mathscr{L}$, $s(a) = 1$ is in fact equivalent to $\pi_k(a)x_n(k) = x_n(k)$, $\forall n$, $\forall k \in K$, since the component $\pi_k(a)$ of a in $\mathscr{L}_k$ is a projection. An analogous relation holds for s'. Hence we have that $\mathscr{I}_s = \mathscr{I}_{s'}$ is equivalent to

$$\pi_k(a)\,x_n(k) = x_n(k), \qquad \forall n, \forall k \in K \Leftrightarrow \pi_k(a)\,y_m(k) = y_m(k), \qquad \forall m, \qquad \forall k \in K \tag{4.4}$$

where a is an element of $\mathscr{L}$. Moreover, it is easy to show that the equalities $\underset{n}{\sigma}\{x_n(k)\} = \underset{m}{\sigma}\{y_m(k)\}$, $\forall k \in K$, imply that the relation (4.4) holds. Conversely, if the relation (4.4) holds true then we get the equalities $\underset{n}{\sigma}\{x_n(k)\} = \underset{m}{\sigma}\{y_m(k)\}$, $\forall k \in K$, taking a in (4.4) first such that the range of $\pi_k(a)$ is $\underset{n}{\sigma}\{x_n(k)\}$, $\forall k \in K$, and then such that the range of $\pi_k(a)$ is $\underset{m}{\sigma}\{y_m(k)\}$, $\forall k \in K$. Hence the relation (4.4) is equivalent to $\underset{n}{\sigma}\{x_n(k)\} = \underset{m}{\sigma}\{y_m(k)\}$, $\forall k \in K$, and this completes the proof of (4.3).

We can now prove that characteristic and pure states coincide.

Proposition 4.1. If s is a state in the Hilbert model, then the following conditions are equivalent:

(a) There are an index $k' \in K$ and a vector $x \in \mathscr{H}_{k'}$, with $\|x\| = 1$, such that $s(a) = (x, \pi_{k'}(a)x)$, $\forall a \in \mathscr{L}$;
(b) s is a characteristic state.

Proof. (a) $\Rightarrow$ (b): take a state s' such that $\mathscr{I}_{s'} = \mathscr{I}_s$. Then, because of (4.3), s' is defined by a sequence of vector fields $\{y_n\}$ such that $\underset{n}{\sigma}\{y_n(k)\} = \{0\}$ for $k \neq k'$ and $\underset{n}{\sigma}\{y_n(k')\} = \sigma\{x\}$. Hence in $\{y_n\}$ there is in fact only one vector field y different from the null one and it has the components $y(k) = 0$ if $k \neq k'$ and $y(k') = \alpha x$, where α is a complex number different from zero. From condition (4.1) it follows that $|\alpha| = 1$. Therefore, s and s' coincide, whence s is characteristic. (b) $\Rightarrow$ (a): if $\{x_n\}$ is a sequence of vector fields which is related to the state s as in (4.2) and $\{y_m\}$ a sequence related to a state s', from (4.3) it follows that the relation

$$\underset{n}{\sigma}\{x_n(k)\} = \underset{m}{\sigma}\{y_m(k)\}, \qquad \forall k \in K \Rightarrow s = s' \tag{4.5}$$

must hold. Let us now suppose that two pairs of indexes (n_1, k_1) and (n_2, k_2) exist such that $x_{n_1}(k_1) \neq 0$ and $x_{n_2}(k_2) \neq 0$. If α and β are two non-null complex numbers such that $|\alpha| \neq 1$ and

$$|\alpha|^2\,\|x_{n_1}(k_1)\|^2 + |\beta|^2\,\|x_{n_2}(k_2)\|^2 = \|x_{n_1}(k_1)\|^2 + \|x_{n_2}(k_2)\|^2$$

(there are infinitely many pairs of numbers for which this property holds true), construct a sequence $\{y_n\}$ in this way:

$$\begin{aligned} y_n(k) &= x_n(k) \qquad \text{if } (n,k) \neq (n_1,k_1),\ (n_2,k_2) \\ y_{n_1}(k_1) &= \alpha x_{n_1}(k_1) \\ y_{n_2}(k_2) &= \beta x_{n_2}(k_2) \end{aligned}$$

Since for $\{y_n\}$ the relation (4.1) holds, $\{y_n\}$ defines a state s' which is different from s, since $s(a) = \|x_{n_1}(k_1)\|^2$ while $s'(a) = |\alpha|^2 \|x_{n_1}(k_1)\|^2$ for $a \in \mathscr{L}$ such that $\pi_k(a)$ is the null projection in $\mathscr{H}_k$ if $k \neq k_1$ and $\pi_{k_1}(a)$ is the one-dimensional projection with range $\sigma\{x_{n_1}(k_1)\}$. As we have constructed $\{y_n\}$ in such a way that $\underset{n}{\sigma}\{x_n(k)\} = \underset{n}{\sigma}\{y_n(k)\}$, $\forall k \in K$, the condition (4.5) fails to be true and, if s has to be characteristic, only one pair of indexes (n,k) must exist for which $x_n(k) \neq 0$. As a consequence, condition (b) holds and the proof of the theorem is complete.

It is now easy to show that, as one could easily expect, in the Hilbert model everything works well, in the sense that the condition (b) of Proposition 3.4 is fulfilled for any element $b \in \mathscr{L}$ different from ϕ. For such an element take in fact an index $k' \in K$ for which the component $\pi_{k'}(b)$ of b is different from the zero projection, a vector $x \in \mathscr{H}_{k'}$, in the range of $\pi_{k'}(b)$ such that $\|x\| = 1$, and construct the characteristic state s defined by $s(a) = (x, \pi_{k'}(a)x)$, $\forall a \in \mathscr{L}$. It is then trivial to see that s is a state of $S_1(b)$.

We have now shown that in the Hilbert model the condition of the existence of 'sufficiently many' characteristic states trivially holds (in the Hilbert model $\mathscr{L}$ is in fact an atomic lattice). However, it has been shown that, if some 'geometric properties' of a proposition system are considered, then 'the physical reality could be too complex in order to fit in any Hilbert space' (Mielnik, 1968). This provides a possible motivation to look for a condition for the atomicity of $\mathscr{L}$ in a picture more general than the Hilbert one.

Finally, it should be noticed that in the Hilbert model we have a bijection between characteristic states and maximal ideals. If p is in fact an atom, then every component except one is zero and the non-null component is a one-dimensional projection. Hence $S_1(p)$ contains exactly the pure state which is represented by the ray corresponding to the range of the non-null component of p. The mapping from the set of characteristic states into the set of maximal ideals which sends s into $\mathscr{I}_s$ is then easily proved to be a bijection, as a consequence of Proposition 3.2 and Proposition 3.3.

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References

Amemiya, I. and Araki, H. (1967). *Publications of the Research Institute of Mathematics and Science, Kyoto University*, **A2**, 423.
Birkhoff, G. (1967). *Lattice Theory*. American Mathematical Society, Providence.
Birkhoff, G. and von Neumann, J. (1936). *Annals of Mathematics*, **37**, 823.
Cirelli, R. and Gallone, F. (1972). *Algebra of Observables and Quantum Logic*. Preprint IFUM 136/FT, University of Milano.
Gallone, F. and Manià, A. (1971). *Annales de l'Institut Henri Poincaré*, **15**, 37.
Gleason, A. M. (1957). *Journal of Mathematics and Mechanics*, **6**, 885.
Gudder, S. and Piron, C. (1971). *Journal of Mathematical Physics*, **12**, 1583.
Gunson, J. (1967). *Communications in Mathematical Physics*, **6**, 262.
Haag, R. and Kastler, D. (1964). *Journal of Mathematical Physics*, **5**, 848.
Jauch, J. M. (1968). *Foundations of Quantum Mechanics*. Addison Wesley.
Jauch, J. M. and Piron, C. (1969). *Helvetica Physica Acta*, **42**, 842.
Kadison, R. V. (1965). *Topology*, **3**, suppl. 2, 177.
Mielnik, B. (1968). *Communications in Mathematical Physics*, **9**, 55.
Piron, C. (1964). *Helvetica Physica Acta*, **37**, 439.
Plymen, R. J. (1968a). *Communications in Mathematical Physics*, **8**, 132.
Plymen, R. J. (1968b). *Helvetica Physica Acta*, **41**, 69.
Pool, J. C. T. (1968). *Communications in Mathematical Physics*, **9**, 118.
Segal, I. E. (1947). *Annals of Mathematics*, **48**, 930.
Varadarajan, V. S. (1968). *Geometry of Quantum Theory*, Vol. **1**. Van Nostrand, Princeton.